Closing Fri: 3.5(1)(2)
Closing *Tues*: 3.6-9
Closing next *Thur*: 3.9

***Entry Task***: Consider $y^3 + x^2 = 4$.

1. Find $\dfrac{dy}{dx}$
2. Find $\dfrac{d^2y}{dx^2}$

1

$$3y^2\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad 3y^2\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{-2x}{3y^2}}$$

2

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(-\frac{2x}{3y^2}\right) \quad \begin{matrix}\leftarrow N \\ \leftarrow D\end{matrix}$$

$$= \frac{(3y^2)(-2) - (-2x)(6y\frac{dy}{dx})}{(3y^2)^2}$$

$$= \frac{-6y^2 + 12xy\frac{dy}{dx}}{9y^4}$$

$$\boxed{\begin{aligned} &= \frac{-6y^2 + 12xy\left(\frac{-2x}{3y^2}\right)}{9y^4} \\ &= \frac{-6y^2 - 8x^2/y}{9y^4} \\ &= \frac{-6y^3 - 8x^2}{9y^5} \end{aligned}}$$

## 3.6 Logarithmic Derivatives

*Recall logarithm facts*:

1. $y = \ln(x) \quad \leftrightarrow \quad e^y = x$

   $y = \log_a(x) \leftrightarrow a^y = x$

   So $\ln(x) = \log_e(x)$

2. $e^{\ln(x)} = x$ and $\ln(e^y) = y$

   $a^{\log_a(x)} = x$ and $\log_a(a^y) = y$

3. $\ln(ab) = \ln(a) + \ln(b)$

   $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

   $\ln(x^n) = \text{n} \ln(x)$

<u>Test of basic understanding</u>

a) Solve $3^x + 1 = 11$

$-1$

$3^x = 10$

$x = \log_3(10)$

$\approx 2.095903$

$\ln(3^x) = \ln(10)$

$x \ln(3) = \ln(10)$

$x = \frac{\ln(10)}{\ln(3)}$

$\approx 2.095903$ CHECK!

b) Solve $(\log_4(2x) - 1)^3 = 8.$

$(\ )^{1/3}$ $\quad \log_4(2x) - 1 = 2$

$+1$ $\quad \log_4(2x) = 3$

$4^{(\ )}$ $\quad 2x = 4^3 = 64$

$\div 2$ $\quad x = \frac{64}{2} = 32$

CHECK

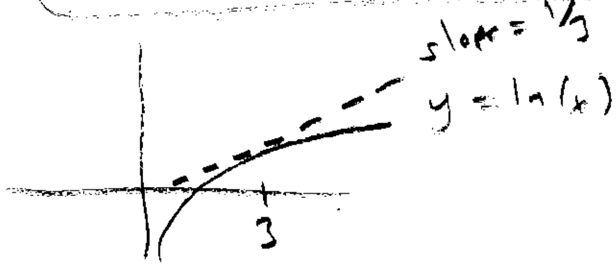
Find the derivative of $y = \ln(x)$

$$e^y = x$$

$$\Rightarrow e^y \frac{dy}{dx} = 1$$

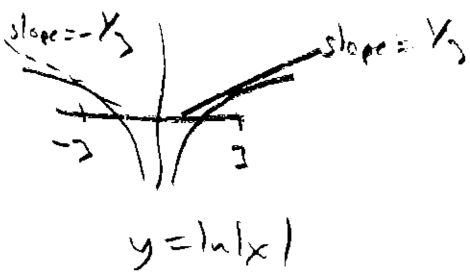
$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\Rightarrow \boxed{\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}}$$




MORE GENERALLY

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$




Find the derivative of $y = \log_a(x)$

$$a^y = x$$

$$\Rightarrow a^y \ln(a) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a^y \ln(a)}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x\ln(a)}}$$

*Example:* Find the derivative of

a) $y = \ln(x^2 - 3x)$

$$\frac{dy}{dx} = \frac{1}{x^2 - 3x} \cdot (2x - 3)$$

$$= \boxed{\frac{2x - 3}{x^2 - 3x}}$$

b) $y = \underbrace{\tan^{-1}(2x)}_{F}\,\underbrace{\ln(3x + 1)}_{S}$

$$\frac{dy}{dx} = \tan^{-1}(2x)\,\frac{1}{3x+1} \cdot 3 + \frac{1}{1+(2x)^2} \cdot 2\,\ln(3x+1)$$

$$= \boxed{\frac{3\tan^{-1}(2x)}{3x+1} + \frac{2\ln(3x+1)}{1+4x^2}}$$

***Power functions***:

$$\frac{d}{dx}\left[\left(g(x)\right)^{n}\right] = n\left(g(x)\right)^{n-1}g'(x)$$

CONSTANT

VARIABLE IN BASE

*Example*:

$$\frac{d}{dx}[(x^3 + 2x)^{10}] =$$

$$= 10\,(x^3+2x)^9\cdot(3x^2+2)$$

***Exponential functions***:

VARIABLE IN EXPONENT

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}g'(x)$$

$$\frac{d}{dx}\left[a^{g(x)}\right] = a^{g(x)}\ln(a)g'(x)$$

CONSTANT

Examples:

$$\frac{d}{dx}\left[e^{(x^4-5x)}\right] = e^{(x^4-5x)}\cdot(4x^3-5)$$

$$\frac{d}{dx}\left[7^{(x^4-5x)}\right] = 7^{(x^4-5x)}\ln(7)\cdot(4x^3-5)$$

**What if *x* is in base AND exponent?**

Example: $y = (3x+1)^x$

Answer: *Logarithmic Differentiation*

*Step 1*: Take log of both sides

*Step 2*: Differentiate implicitly

*Step 3*: Solve for y'.

$$\ln(y) = \ln\left((3x+1)^x\right)$$

$$\Rightarrow \ln(y) = x\ln(3x+1)$$

or WRITE $y = e^{x\ln(3x+1)}$

$$\frac{dy}{dx} = e^{x\ln(3x+1)} \cdot \left(\frac{3x}{3x+1} + \ln(3x+1)\right)$$

$$\boxed{= (3x+1)^x \left(\frac{3x}{3x+1} + \ln(3x+1)\right)}$$

$$\frac{1}{y}\frac{dy}{dx} = x\,\frac{1}{3x+1}\cdot 3 + (1)\ln(3x+1)$$

$$\frac{dy}{dx} = y\left(\frac{3x}{3x+1} + \ln(3x+1)\right)$$

$$\boxed{= (3x+1)^x\left(\frac{3x}{3x+1} + \ln(3x+1)\right)}$$

*Example* (Directly from HW):

Find $dy/dx$.

$$y = (\sin(7x))^{\ln(x)}$$

$$\ln(y) = \ln(x)\ln(\sin(7x))$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\sin(7x)) + \ln(x)\frac{1}{\sin(7x)}\cos(7x)\cdot 7$$

$$\boxed{\frac{dy}{dx} = y\left(\frac{\ln(\sin(7x))}{x} + 7\ln(x)\cot(7x)\right)}$$

# Preview of 3.9 (Related Rates)

*Example* (from homework):

If a (spherical) snowball melts so that its surface area decreases at a rate of 5 cm²/min, find the rate at which the diameter decreases when the diameter is 12 cm.

## Steps to all these problems

1. Draw and label a picture
2. What do you ***know***?

   What do you ***want***?
3. Write equations relating quantities.
4. Differentiate to get rates.
5. Plug in values (***wait*** until end)

$r = r(t)$
$d = d(t)$
$V = V(t)$
$S = S(t)$

$r$ = radius — cm
$D$ = diameter — cm
$V$ = volume $= \frac{4}{3}\pi r^3$ — cm³
$S$ = surface area $= 4\pi r^2$ — cm²

GIVEN $\frac{dS}{dt} = -5$

WANT $\frac{dD}{dt} = ??$ when $D = 12$

① $D = 2r \Rightarrow \frac{dD}{dt} = 2\frac{dr}{dt}$

② $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

PLUG IN $\frac{dS}{dt} = -5$, $D = 12$

$D = 12 \Rightarrow r = 6$

$\frac{dS}{dt} = -5 \Rightarrow -5 = 8\pi(6)\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = -\frac{5}{48\pi}$

$\Rightarrow \frac{dD}{dt} = 2\left(-\frac{5}{48\pi}\right)$

$\boxed{= -\frac{5}{24\pi}\ \frac{\text{cm}}{\text{min}} \approx -0.0663\ \frac{\text{cm}}{\text{min}}}$